Limit theorems for a class of critical superprocesses with stable branching

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This talk is based on joint works with Renming Song and Zhenyao Sun

Outline

1. Limit results under second moment conditions
   - Critical G-W processes
   - General critical branching processes
   - Critical superprocesses
2. Limit results without second moment conditions
   - Critical G-W processes
   - Critical continuous-state branching processes
3. Superprocesses with stable branching mechanism
   - Introduction of superprocesses
   - Main results
   - Technique: Size-biased transform of Poisson random measures

Limit results under second moment conditions

Branching process

- Let L be an $\mathbb{N}_0$-valued random variable, $EL = m$, $\mathrm{Var}(L) = \sigma^2$.
- Consider a branching particle system such that:
  - There is one particle at generation 0.
  - Each particle in the system independently produces a random number of new particles, according to L.
  - The reproduction goes recursively.
- Denote by Z_n the number of particles at generation n, then we say the process $(Z_n)_{n\geq 1}$ is a **Galton-Watson process**.
- It is well known that

$$\lim_{n\to\infty} P(Z_n > 0) = P(\forall n \; s.t. \; Z_n > 0) = 0,$$

iff $m \leq 1$.

Kolmogorov's and Yaglom's results

When the branching process $(Z_n, n \geq 1)$ is **critical**, i.e. $m = 1$, and $\mathrm{Var}(L) = \sigma^2 < \infty$,

- Kolmogorov (1938) proved that

$$nP(Z_n > 0) \xrightarrow[n\to\infty]{} \frac{2}{\sigma^2} \tag{1}$$

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$$\left\{\frac{Z_n}{n}; \quad P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{law} \frac{\sigma^2}{2}\mathbf{e}, \tag{2}$$

where **e** is an exponential random variable with mean 1.

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- We will call results like (1) Kolmogorov type results and results like (2) Yaglom type results on more general branching processes.

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- **critical branching Markov processes**, see Asmussen and Hering (1983).

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- **critical branching Markov processes**, see Asmussen and Hering (1983).
- **critical superprocesses**, see Evans and Perkins (1990) and R., Song and Zhang (2015).

General critical branching processes (Probabilistic proofs)

- **Kolmogorov**'s result and **Yaglom**'s result on **branching process** see Lyons, Pemantle and Peres (1995),Geiger (1999), and R., Song and Sun (2018).

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- **Kolmogorov** type and **Yaglom** type results on **branching diffusions**, see Powell (2015).
- **Kolmogorov** type and **Yaglom** type results for a class of critical superprocesses, see R., Song and Sun (2017).

Superprocesses

Superprocesses are measura-valued Markov processes. To define it, we need some preparation:

- E: locally compact separable metric space with a measure m.
- $\mathcal{M}_f$: the collection of all the finite Borel measures on E.
- (ξ_t): an E-valued Hunt process with transition semigroup (P_t).
- $\Psi : E \times [0, \infty) \to [0, \infty)$ s.t.

$$\Psi(x, z) := -\beta(x)z + \alpha(x)z^2 + \int_{(0,\infty)} (e^{-zy} - 1 + zy)\pi(x, dy).$$

  where

  - $\beta \in b\mathscr{B}_E$;
  - $\alpha \in bp\mathscr{B}_E$;
  - π: a kernel from E to $(0, \infty)$ s.t. $\sup_{x\in E} \int_{(0,\infty)} (y \wedge y^2)\pi(x, dy) < \infty$.

Superprocesses

- $\mu(f) := \int_E f(x)\mu(dx), \quad f \in b\mathscr{B}_E, \mu \in \mathcal{M}_f.$

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A $\mathcal{M}_f$-valued Markov process $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ is called to be a (ξ,Ψ)-superprocess if it satisfies that

$$\mathbf{P}_\mu[e^{-X_t(f)}] = e^{-\mu(u_f(t,\cdot))}, \quad t \geq 0, \mu \in \mathcal{M}_f, f \in bp\mathscr{B}_E.$$

Criticality of Superprocesses

- $(S_t)_{t\geq 0}$: mean semigroup of superprocess (X_t) defined by

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- It is known that $(e^{-\lambda t} X_t(\phi))_{t\geq 0}$ is a nonnegative martingale.
- When $\lambda = 0$ $(> 0, \ < 0)$, we say the process is (super, sub) critical.

Limit results on critical superprocesses

Now let us consider critical superprocess (X_t). Under some other conditions, it was proved by R., Song and Zhang (2015), and R., Song and Sun (2017) that

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- Here, the constant $c_0 > 0$ is independent of the choice of μ and f.

Limit results without second moment conditions

Galton-Watson process

Suppose $(Z_n)_{n\geq 1}$ is a **Galton-Watson process**. $\mathrm{Var}(L) = \sigma^2 = \infty$

- Zolotarev (1957) and Slack (1968): Assume that the generating function $f(s)$ of the offspring distribution is of the form

$$f(s) = s + (1-s)^{1+\alpha} l(1-s), \quad s \geq 0, \tag{3}$$

where $\alpha \in (0, 1]$ and l is a function slowly varying at 0. Then

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  - $$P(Z_n > 0) = n^{-1/\alpha} L(n), \tag{4}$$

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  - $$\left\{P(Z_n > 0)Z_n;\ P(\cdot|Z_n > 0)\right\} \xrightarrow[n\to\infty]{\text{law}} \mathbf{z}^{(\alpha)}, \tag{5}$$

    where $\mathbf{z}^{(\alpha)}$ is a positive random variable with Laplace transform

$$E[e^{-u\mathbf{z}^{(\alpha)}}] = 1 - (1+u^{-\alpha})^{-1/\alpha}, \quad u \geq 0. \tag{6}$$

Galton-Watson process

- Slack (1972) considered the converse of this problem: In order for $\{P(Z_n > 0)Z_n;\ P(\cdot|Z_n > 0)\}$ to have a non-degenerate weak limit, the generating function of the offspring distribution must be of the form of (3) for some $0 < \alpha \leq 1$.

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- For shorter and more unified approaches to these results, we refer our readers to Borovkov (1989) and Pakes (2010).

More general critical branching processes

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- It is natural to ask whether similar results are still valid for some **critical superprocesses** without the second moment condition.

Continuous-state branching processes(CSBPs)

- Kyprianou and Pardo (2008) considered CSBPs $\{(Y_t)_{t\geq 0}; P\}$ with stable branching mechanism $\psi(z) = cz^\gamma$ where $c > 0$ and $\gamma \in (1, 2]$. For all $x > 0$, with $c_t := (c(\gamma - 1)t)^{1/(\gamma-1)}$,

$$\{c_t^{-1} Y_t; P(\cdot | Y_t > 0, Y_0 = x)\} \xrightarrow[t\to\infty]{\text{law}} \mathbf{z}^{(\gamma-1)}. \tag{7}$$

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- R., Yang and Zhao (2014) studied CSBPs with branching mechanism

$$\psi(z) = cz^2 l(z), \quad z \geq 0, \tag{8}$$

where $c > 0$, $\gamma \in (1, 2]$ and l is a function slowly varying at 0. For all $x > 0$, with $\lambda_t := P_1(Y_t > 0)$,

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- Iyer, Leger and Pego (2015) considered the converse problem: Suppose the branching mechanism ψ satisfies Grey's condition. In order for the left side of (9) to have a non-trivial weak limit for some positive constants $(\lambda_t)_{t\geq 0}$, one must have (8) for some $1 < \gamma \leq 2$.

Superprocesses with stable branching mechanism

Settings

- E: locally compact separable metric space.
- $\mathcal{M}_f$: the collection of all the finite Borel measures on E.
- **Spatial motion** (ξ_t): an E-valued Hunt process with transition semigroup (P_t) and lifetime ζ.

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$$\psi(x,z) = -\beta(x)z + \kappa(x)z^{\gamma(x)}, \quad x \in E, z \geq 0, \tag{10}$$

where $\beta \in \mathscr{B}_b(E)$, $\gamma \in \mathscr{B}_b^+(E)$, $\kappa \in \mathscr{B}_b^+(E)$ with $1 < \gamma(\cdot) < 2$, $\gamma_0 := \operatorname{ess\,inf}_{m(dx)} \gamma(x) > 1$ and $\operatorname{ess\,inf}_{m(dx)} \kappa(x) > 0$.

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- $(X_t)_{t\geq 0}$: a superprocess with spatial motion ξ and branching mechanism Ψ.

Superprocesses

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Definition (Superprocesses)

A $\mathcal{M}_f$-valued Markov process $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ is called to be a (ξ,Ψ)-superprocess if it satisfies that

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- If $E = \{x_0\}$ then $Z_t := X_t(1)$ is simply a CSBP.

Assumptions

- The mean behavior of superprocess can be described by the Feynman-Kac transform of (P_t):

$$\mathbf{P}_{\delta_x}[X_t(f)] = P_t^{\beta} f(x) := \Pi_x[e^{\int_0^t \beta(\xi_r)dr} f(\xi_t)\mathbf{1}_{t<\zeta}],$$

for $x \in E, t \geq 0, f \in b\mathscr{B}_E$.

Assumption 1. (Compact operators)

There exist a σ-finite Borel measure m with full support on E and a family of strictly positive, bounded continuous functions $\{p_t(\cdot,\cdot) : t > 0\}$ on $E \times E$ such that,

- $P_t f(x) = \int_E p_t(x,y)f(y)m(dy), \quad t > 0, x \in E, f \in b\mathscr{B}_E,$
- $\int_E p_t(y,x)m(dy) \leq 1, \quad t > 0, x \in E,$
- $\int_E \int_E p_t(x,y)^2 m(dx)m(dy) < \infty, \quad t > 0.$
- $x \mapsto \int_E p_t(x,y)^2 m(dy)$ and $x \mapsto \int_E p_t(y,x)^2 m(dy)$ are both continuous on E.

Assumptions

- $(P_t^\beta)_{t\geq 0}$ and its disjoint semigroup $(P_t^{\beta*})_{t\geq 0}$ are both strongly continuous semigroups of compact operators in $L^2(E, m)$.
- L and L^*: the generators of $(P_t^\beta)_{t\geq 0}$ and $(P_t^{\beta*})_{t\geq 0}$, respectively.
- $\lambda := \sup \mathrm{Re}(\sigma(L)) = \sup \mathrm{Re}(\sigma(L^*))$, a common eigenvalue of multiplicity 1.
- ϕ and ϕ^*: the eigenfunction of L and L^* associated with the eigenvalue λ.
- Normalize ϕ and ϕ^* by $\langle \phi, \phi \rangle_m = \langle \phi, \phi^* \rangle_m = 1$.

Assumption 2. (Critical and Intrinsic Ultracontractive)

- $\lambda = 0$. (Critical)
- $\forall t > 0, \exists c_t > 0, \forall x, y \in E, \quad p_t^\beta(x, y) \leq c_t \phi(x) \phi^*(y)$. (Intrinsic Ultracontractive)

Main results

Theorem

Suppose that $\{(X_t)_{t\geq 0}; (\mathbf{P}_\mu)_{\mu\in\mathcal{M}_f}\}$ *is a* (ξ,ψ)*-superprocess satisfying Assumptions 1-2. Then,*

(1) *For each* $t > 0$ *and* $x \in E$, $\mathbf{P}_{\delta_x}(\|X_t\| = 0) > 0$.

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(2) *For each* $\mu\in\mathcal{M}^1_E$, $\mathbf{P}_\mu(\|X_t\|\neq 0)$ *converges to* 0 *as* $t\to\infty$ *and is regularly varying at infinity with index* $-(\gamma_0-1)^{-1}$. *Furthermore, if* $m(x:\gamma(x)=\gamma_0)>0$, *then*

$$\lim_{t\to\infty}\eta_t\mathbf{P}_\mu(\|X_t\|\neq 0)=\mu(\phi). \tag{11}$$

Here, $\eta_t := \left(C_X(\gamma_0-1)t\right)^{\frac{1}{\gamma_0-1}}$, $C_X := \langle \mathbf{1}_{\gamma(\cdot)=\gamma_0}\kappa\phi^{\gamma_0},\phi^*\rangle_m$.

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(3) *Suppose* $m(x:\gamma(x)=\gamma_0)>0$*. Let* $f\in\mathscr{B}^+(E)$ *be such that* $\langle f,\phi^*\rangle_m>0$ *and* $\|\phi^{-1}f\|_\infty<\infty$*. Then for each* $\mu\in\mathcal{M}_E^1$,

$$\{\eta_t^{-1}X_t(f);\ \mathbf{P}_\mu(\cdot|\|X_t\|\neq 0)\}\xrightarrow[t\to\infty]{\text{law}}\langle f,\phi^*\rangle_m\mathbf{z}^{(\gamma_0-1)}. \tag{12}$$

Size-biased transformation

- Let $(\Omega, \mathscr{F})$ be a measurable space with a σ-finite measure ν. For any $0 \le F \in \mathscr{F}$ such that $\nu(F) \in (0, \infty)$, we define the *F-transform of ν* as the probability ν^F on $(\Omega, \mathscr{F})$ such that

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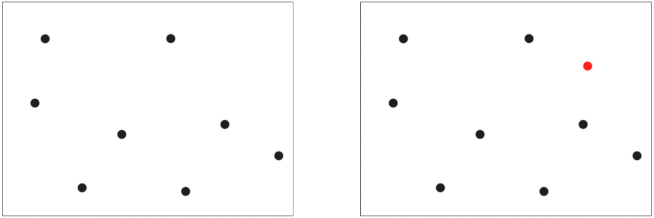
  - Let $\{(X_t)_{t\in\Gamma}; P\}$ be a stochastic process. A process $\{(\dot{X}_t)_{t\in\Gamma}; \dot{P}\}$ is called a F-transform of process (X_t) if $\{(\dot{X}_t)_{t\in\Gamma}; \dot{P}\} \overset{f.d.d.}{=} \{(X_t)_{t\in\Gamma}; P^F\}$.

Size-biased transform of Poisson random measures

- $\mathcal{N}$: a Poisson random measure on a measurable space $(S, \mathscr{S})$ with intensity measure N.
- $F \in \mathscr{S}^+$: $0 < N(F) < \infty$ (, which implies that $P(\mathcal{N}(F)) < \infty$).

Theorem (R., Song and Sun (2017))

$$\{\mathcal{N}; P^{\mathcal{N}(F)}\} \stackrel{d}{=} \{\mathcal{N} + \delta_s; P \otimes N^F(ds)\}.$$



Superprocesses as PRMs

- $\mathcal{W}$: Skorokhod space of $\mathcal{M}_f$-valued càdlèg paths.
- $(\mathbb{N}_x)_{x\in E}$: Kuznetsov measure (N-measure, excursion measure) of superprocess (X_t).
- $\mu \in \mathcal{M}_f$.
- $\mathcal{N}_\mu$: a Poisson random measure on $\mathcal{W}$ with intensity measure

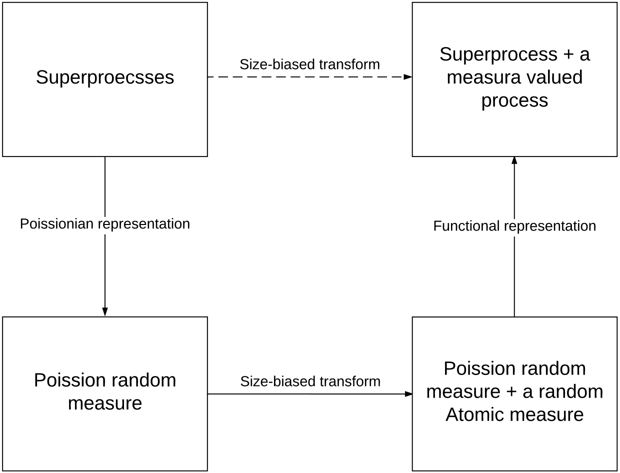
$$\int_E \mathbb{N}_x[\,\cdot\,]\mu(dx).$$

Theorem (Superprocesses as PRMs, see Li (2011) Theorem 8.24)

$$\{(X_t)_{t>0}; \mathbf{P}_\mu\} \stackrel{d}{=} \left(\int_{\mathcal{W}} w_t \mathcal{N}_\mu(dw)\right)_{t>0},$$

here $(w_t)_{t\geq 0}$ is the coordinate process.

Idea


Superproecsses
Size-biased transform
Superprocess + a measura valued process
Poissionian representation
Functional representation
Poission random measure
Size-biased transform
Poission random measure + a random Atomic measure

Size-biased transforms of superprocesses

- F: a non-negative measurable function on $\mathcal{W}$ s.t. $\mathbb{N}_\mu[F] \in (0, \infty)$.

Theorem (R., Song and Sun (2017) [18])

$$\{(X_t)_{t\geq 0}; \mathbf{P}_\mu^{\mathcal{N}(F)}\} \overset{d}{=} \{(X_t + w_t)_{t\geq 0}; \mathbf{P}_\mu \otimes \mathbb{N}_\mu^F(dw)\}.$$

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  - $F(w) = w_t(\phi)^2$ using a 2-Spine Decomposition Theorem.
  - ......

1-spine decomposition, $F(w) := w_T(g)$

Let $g \in p\mathscr{B}_E$ s.t. $\|\phi^{-1}g\|_\infty < \infty$. Let $\mu \in \mathcal{M}_f$ s.t $\mu(\phi) < \infty$. Let $T > 0$. We can constract a $\mathcal{M}_f$-valued process $\{(Y_t)_{0\le t\le T}; \ddot{\mathbf{P}}_\mu^{(T,g)}\}$ which is a realization of $w_T(g)$-transform of $\mathbb{N}_x$:

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- Let the **spine process** $\{\xi; \dot{\mathbf{P}}_x^{(T,g)}\}$ be a $\left(g(\xi_T)e^{\int_0^T \beta(\xi_s)ds}\right)$-transform of $\{\xi; \mathbb{P}_x\}$.

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- Conditioned on ξ, let $\{\mathbf{n}; \dot{\mathbf{P}}_x^{(T,g)}\}$ be a Poisson random measure on $(0, T] \times \mathcal{W}$ with mean measure

$$2\alpha(\xi_s)ds \times \mathbb{N}_{\xi_s}(dw) + ds \times \int_{(0,\infty)} y\mathbf{P}_{y\delta_{\xi_s}}(dw)\pi(\xi_s, dy).$$

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- The **immigration along the spine** $\{Y; \dot{\mathbf{P}}_x^{(T,g)}\}$ is an $\mathcal{M}_f$-valued process defined by

$$Y_t(\cdot) := \int_{(0,t]\times\mathcal{W}} w_{t-s}(\cdot)\mathbf{n}(ds, dw), \quad t \ge 0.$$

1-spine decomposition

Theorem (R., Song and Sun (2017))

Let (Y_t) be the spine immigration defined above. Then $(Y_t)_{0\le t\le T}$ is the $w_T(g)$-transform of the Kuznetsov measure $\mathbb{N}_\mu$.

When $g = \phi$, the above result degenerates to the classical spine decomposition theorem developed by Eckhoff, Kyprianou and Winkel (2015), Engländer and Kyprianou (2004), and Liu, R. and Song (2009).

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END

Thank you!

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